MAT 1332: Calculus for Life Sciences

A course based on the book
Modeling the dynamics of life
by F.R. Adler

Supplementary material
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The inverse tangent function

The tangent function is defined as

$$\tan(x) = \frac{\sin(x)}{\cos(x)},$$

and its derivative can be compute by the quotient rule as

$$\frac{d}{dx}\tan(x) = \frac{\frac{d}{dx}\sin(x)\cos(x) - \sin(x)\frac{d}{dx}\cos(x)}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}.$$

In particular, the function is defined for all x that are not odd multiples of π, and the function is monotone increasing, see Figure 1.

The inverse of the tangent is denoted as arctan or $\tan^{-1}$ and it is defined in the usual way as

$$\arctan(\tan(x)) = x, \qquad \tan(\arctan(x)) = x,$$

see Figure 1. What is its derivative? We differentiate the first equality above by the chain rule and find

$$\frac{d}{dx}[\arctan(\tan(x))] = \frac{d}{dy}\arctan(y)\frac{d}{dx}\tan(x) = 1,$$

with $y = \tan(x)$, since $\frac{d}{dx}x = 1$. Hence, we can divide

$$\frac{d}{dy}\arctan(y) = \frac{1}{\frac{d}{dx}\tan(x)} = \frac{\cos^2(x)}{\cos^2(x)+\sin^2(x)} = \frac{1}{1+\frac{\sin^2(x)}{\cos^2(x)}} = \frac{1}{1+\tan^2(x)} = \frac{1}{1+y^2}.$$

Application to integration

We can now integrate the derivative of the arctan function to get

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C.$$

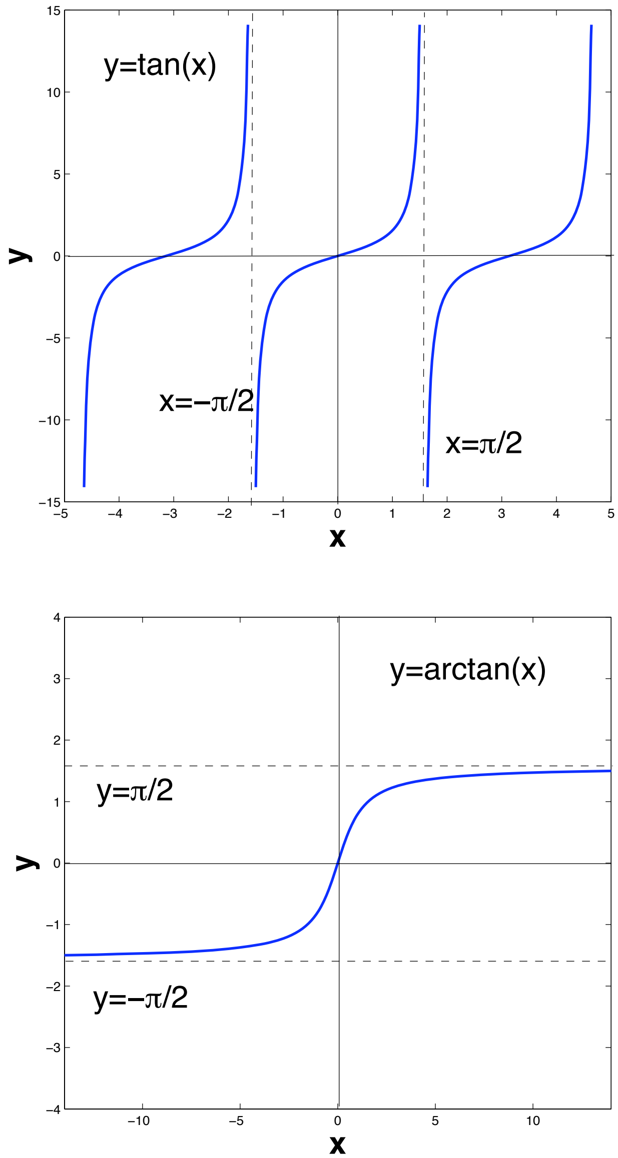

Figure 1: Graphs of the tangent function and its inverse, the arctangent function

Integrals and volumes

First introductory example

The *E. coli* bacterium has a rod-like shape with a cylinder in the middle and two half balls at the ends. What is its volume?
The volume of a cylinder with radius r and height h is $\pi r^2 h$. The volume of a ball with radius r is $4\pi r^3/3$. Now we only have to add the two volumes. For *E. coli*, the data are approximately $r = 0.8 * 10^{-6}$m and $h = 2 * 10^{-6}$m. This gives a volume of $V = \pi(1.28 + 0.6825)\text{m}^{-18} = 6.16\text{m}^{-18}$.

Second introductory example

A termite mount is approximately cone shaped. What is its volume? If we imagine that we cut a cone into thin slices (perpendicular to its rotational axis), then each slice is approximately a cylinder. For each of these cylinders, we know how to compute the volume; and then we add the volumes. This procedure is very similar to the Riemann sums for the area under a curve, except that we are now using three-dimensional objects. But it gives us the right idea.

General Idea

To calculate the volume of an object with rotational symmetry, we need to know the diameter or radius at each point of its rotational axis. This gives the area of the cut surface at each point. Then we find the volume by integrating the area, see Figure 2.

More precisely, assume that the rotational axis is the x-axis and the radius at each point is given by $f(x)$. Then the area of the cut surface at point x is $A(x) = \pi f^2(x)$ and the volume of the object between points a and b is given by

$$\text{Volume} = V = \int_a^b A(x)dx = \pi \int_a^b (f(x))^2 dx.$$

Example 1: The cylinder

Rotating a constant function $f(x) = r$ around the x-axis gives a cylinder. The volume of the cylinder between $x = 0$ and $x = h$ is

$$V = \int_0^h \pi r^2 dx = \pi r^2 [x]_0^h = \pi r^2 h.$$

Of course, we knew that before.

Example 2: The ball

Rotating the function $f(x) = \sqrt{r^2 - x^2}$ with $-r \leq x \leq r$ around the x-axis gives a ball of radius r. The volume is given by

$$V = \int_{-r}^r \pi(f(x))^2 dx = \int_{-r}^r \pi(r^2 - x^2)dx = \pi \left[r^2 x - \frac{1}{3}x^3 \right]_{-r}^r = \frac{4}{3}\pi r^3.$$

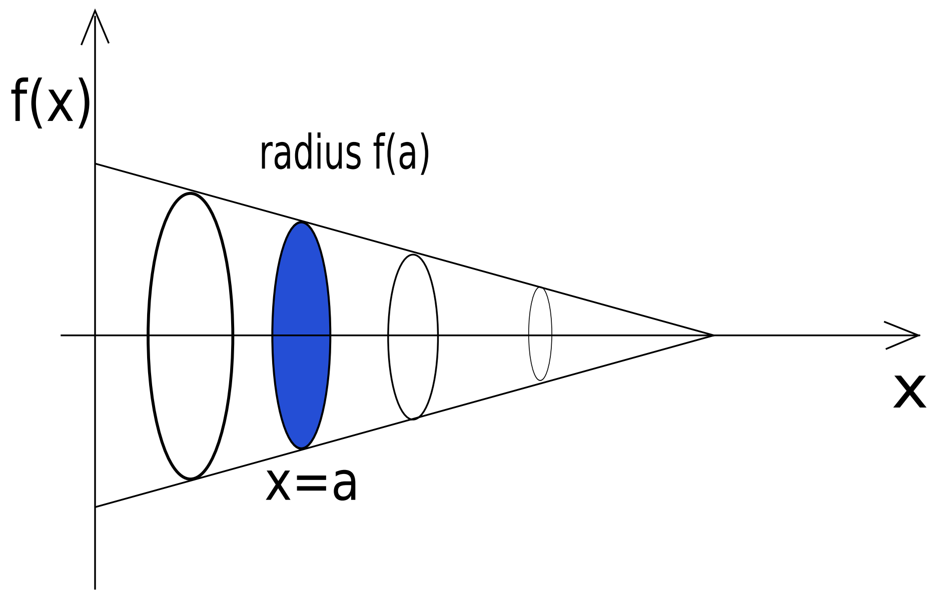

Figure 2: Illustration of volumes of rotation. The straight line function $f(x)$ is rotated around the x-axis. At $x = a$ the radius of the slice is $f(a)$, the area of the slice is $\pi(f(a))^2$. The volume of the cone is given by integrating the area between the base and the top.

Of course, we know that, too. But now we can even compute the volume of a ball with the top and/or bottom removed. For example, what is the volume of the earth (radius $6.3 * 10^3$ km) with the north and south pole removed (say 500 km on either end)?

$$V = \int_h^h \pi(r^2 - x^2)dx = \pi \left[r^2 x - \frac{1}{3}x^3\right]_{-h}^h = \pi \left[2r^2 h - \frac{2}{3}h^3\right].$$

With $r = 6.3 * 10^3$km and $h = 5.8 * 10^3$km we get $V = \pi \left(2(6.3)^2 * 5.8 - \frac{2}{3}(5.8)^3\right) * 10^9\text{km}^3 = \pi(460.4040 - 130.0747) * 10^9\text{km}^3 \approx 10^{10}\text{km}^3$

Example 3: The termite mound

Suppose the cone-shaped termite mount has a base diameter of 1 meter and a hight of 2 meters. What is its volume?
At first, we have to find the radius at any given hight. At the bottom, the radius is 0.5m, at 2m it is zero. In between, we need a linear function. The function $f(x) = 0.5(1 - x/2)$ gives the profile. Then we integrate to get the volume.

$$V = \pi \int_0^2 (f(x))^2 dx = \frac{\pi}{4}\int_0^2 (1-x/2)^2 dx = \frac{\pi}{4}\int_0^2 (1-x+x^2/4)dx = \frac{\pi}{4}[x-x^2/2+x^3/12]_0^2 = \frac{8\pi}{48} \approx 0.52.$$

The units are, of course, cubic meter.

Example 4: The tree

Suppose that a tree trunk is 25 meters high, and that its radius at height x meter is $r(x) = 2\exp(-x)$ meter. What is its volume?

$$V = \pi \int_0^{25} (2\exp(-x))^2 dx = 4\pi \int_0^{25} e^{-2x} dx = 2\pi \left[-e^{-2x}\right]_0^{25} = 2\pi(1 - e^{-50}) \approx 2\pi.$$

The units are, of course, cubic meter.

Example 5: Trig functions

What is the volume obtained by rotating the function $f(x) = \cos\left(\frac{x}{2}\right)$ with $-\pi \le x \le \pi$ around the x-axis? The volume is given by the formula

$$\begin{aligned} V &= \pi \int_{-\pi}^{\pi} [f(x)]^2\, dx \\ &= \pi \int_{-\pi}^{\pi} \cos^2\left(\frac{x}{2}\right) dx\,. \end{aligned}$$

We can't integrate $\cos^2\theta$ directly, so we have to use our long-forgotten-but-recently-looked-up-in-the-textbook trigonemetric identities. Thus

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \qquad\qquad (\text{since } \cos^2\theta + \sin^2\theta = 1) \\ &= 2\cos^2\theta - 1\,. \end{aligned}$$

Thus $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$. Substituting this into the integral, we have

$$V = \pi \int_{-\pi}^{\pi} \frac{\cos 2\left(\frac{x}{2}\right) + 1}{2} dx = \frac{\pi}{2} \int_{-\pi}^{\pi} (\cos x + 1)\, dx = \frac{\pi}{2} \Big[\sin x + x \Big]_{-\pi}^{\pi}$$

$$= \frac{\pi}{2} \left[(\sin\pi + \pi) - (\sin(-\pi) - \pi) \right] = \frac{\pi}{2}(2\pi) = \pi^2 \text{ units}^3 .$$

Partial fractions

First introductory example

We don't know how to integrate the fraction $\int \frac{x}{x+2} dx$, but we can write the fraction in a simpler way and use known rules to find the integral as follows:

$$\int \frac{x}{x+2} dx = \int \frac{(x+2)-2}{x+2} dx = \int \left[1 - \frac{2}{x+2}\right] dx = x - 2\ln(x+2) + C.$$

Second introductory example

We don't know integrate the fraction $\int \frac{3x-2}{x(x-2)} dx$. But if we simplify the fraction as

$$\frac{3x-2}{x(x-2)} = \frac{1}{x} + \frac{2}{x-2}$$

(check this!) then we can integrate as follows:

$$\int \frac{3x-2}{x(x-2)} dx = \int \left(\frac{1}{x} + \frac{2}{x-2}\right) dx = \ln|x| + 2\ln|x-2| + C.$$

General Idea

Rational functions are fractions of polynomials, i.e., if $P(x)$ and $Q(x)$ are polynomials, then $P(x)/Q(x)$ is called a rational function. We already know how to integrate some of them, namely the following building blocks (you need to know these!)

$$\begin{aligned} \int \frac{1}{x+a} dx &= \ln|x+a| + C, \\ \int \frac{1}{x^2+1} dx &= \arctan(x) + C = \tan^{-1}(x) + C, \\ \int \frac{x}{x^2+1} dx &= \frac{1}{2}\ln(x^2+1) + C. \end{aligned}$$

If we can split a rational function into sums of these building blocks, then we can integrate easily. The goal of this section is to find a technique how to integrate (find antiderivatives of) all rational functions. We only consider cases where $\deg(Q) \le 2$, i.e., the highest power of x in the denominator is no more than 2. The idea is to decompose a rational function into a sum of simpler rational functions, namely the three examples above, which we know how to integrate.

Recipe of partial fractions

To find the integral of a rational function $P(x)/Q(x)$ follow these steps.

1. If $\deg(P) \ge \deg(Q)$ then use long division to split the rational function into several parts. Now assume that $\deg(P) < \deg(Q)$.

2. If $Q(x) = ax^2 + bx + c = a(x - x_1)(x - x_2)$ has two distinct real roots, the one can find numbers A, B such that
$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A}{x - x_1} + \frac{B}{x - x_2}\right].$$
Then use the natural logarithm to integrate the two terms.

3. If $Q(x) = ax^2 + bx + c = a(x - x_1)^2$ has only one real root, the one can find numbers A, B such that
$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left[\frac{A}{x - x_1} + \frac{B}{(x - x_1)^2}\right].$$
Then one can integrate using substitution, the logarithm, and direct integration.

4. If $Q(x) = ax^2 + bx + c$ has no real roots, then complete the square to get
$$Q(x) = a\left[\left(x - \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] = a[(x - A)^2 + B].$$
Then use the natural logarithm and the arctan to integrate the two terms (potentially substitute first).

We illustrate each of these cases with examples.

Example 1

$P(x) = x^2 + 1, Q(x) = x - 1$. Then $\deg(P) = 2 > 1 = \deg(Q)$, hence we need to do long division. We find
$$x^2 + 1 = (x - 1)(x + 1) + 2.$$
Therefore
$$\int \frac{x^2 + 1}{x - 1} dx = \int \left[x + 1 + \frac{2}{x - 1}\right] dx = \frac{x^2}{2} + x + 2\ln|x - 1| + C.$$

Example 2

$P(x) = 2x^3 + 3x^2 + 2x + 4, Q(x) = x^2 + 1$. Again, since $\deg(P) = 3 > 2 = \deg(Q)$, hence we need to do long division. We find
$$2x^3 + 3x^2 + 2x + 4 = (x^2 + 1)(2x + 3) + 1.$$
Therefore
$$\int \frac{2x^3 + 3x^2 + 2x + 4}{x^2 + 1} dx = \int \left[2x + 3 + \frac{1}{x^2 + 1}\right] dx = x^2 + 3x + \arctan(x) + C.$$

Example 3

$P(x) = 2x+1, Q(x) = x^2+x-2$. This time, $\deg(P) = 1 < 2 = \deg(Q)$, so no long division necessary. Instead we factor Q as $Q(x) = (x-1)(x+2)$, so that

$$\frac{2x+1}{x^2+x-2} = \frac{2x+1}{(x-1)(x+2)}.$$

On the other hand, for two numbers, A, B, we find

$$\frac{A}{x-1} + \frac{B}{x+2} = \frac{(A+B)x+2A-B}{(x-1)(x+2)}.$$

Comparing with the expression above, we find that $A+B = 2$ and $2A-B = 1$. Hence, $A = B = 1$. Then we integrate

$$\int \left[\frac{2x+1}{x^2+x-2}\right] dx = \int \left[\frac{1}{x-1} + \frac{1}{x+2}\right] dx = \ln|x-1| + \ln|x+2| + C.$$

Example 4

$P(x) = x+5, Q(x) = x^2-4x+4$. Again, $\deg(P) = 1 < 2 = \deg(Q)$, so no long division necessary. But $Q(x) = (x-2)^2$, has only a single root, i.e.,

$$\frac{x+5}{x^2-4x+4} = \frac{x+5}{(x-2)^2}.$$

On the other hand, for two numbers, A, B, we find

$$\frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{Ax-2A+B}{(x-2)^2}.$$

Comparing with the expression above, we find that $A = 1$ and $-2A+B = 5$. Hence, $A = 1, B = 7$. Then we integrate

$$\int \left[\frac{x+5}{x^2-4x+4}\right] dx = \int \left[\frac{1}{x-2} + \frac{7}{(x-2)^2}\right] dx = \ln|x-2| - \frac{7}{x-2} + C.$$

Example 5

$P(x) = 3x+2, Q(x) = x^2-2x+5$. No long division necessary. However, Q has no real roots. We complete the square

$$Q(x) = x^2-2x+5 = x^2-2x+1-1+5 = (x-1)^2+4.$$

Now we write

$$\int \frac{3x+2}{x^2-2x+5} dx = \int \frac{3x+2}{(x-1)^2+4} dx = \frac{1}{4} \int \frac{3x+2}{(\frac{x-1}{2})^2+1} dx.$$

This is a case for substitution. We choose $u = \frac{x-1}{2}$ so that $x = 2u + 1$ and $dx = 2du$. Then we get

$$\frac{1}{4}\int \frac{3x+2}{(\frac{x-1}{2})^2+1}dx = \frac{1}{2}\int \frac{6u}{u^2+1}du + \frac{1}{2}\int \frac{5}{u^2+1}du.$$

The first of these integrals requires another substitution, $w = u^2 + 1$, the second is again an acrtan. With this we find

$$\frac{1}{2}\int \frac{6u}{u^2+1}du + \frac{1}{2}\int \frac{5}{u^2+1}du = \frac{1}{2}\int \frac{3}{w}dw + \frac{1}{2}\int \frac{5}{u^2+1}du = \frac{3}{2}\ln|w| + \frac{5}{2}\arctan(u) + C.$$

After back-substituting we find that the integral with respect to x is given by

$$\frac{3}{2}\ln\left|\frac{x^2}{4} - \frac{x}{2} + \frac{5}{4}\right| + \frac{5}{2}\arctan\left(\frac{x-1}{2}\right) + C.$$

Example 6

$P(x) = x^2 - 2, Q(x) = x^2 - 3x + 2$. Long division first, or the simpler way

$$\frac{x^2-2}{x^2-3x+2} = \frac{x^2-3x+2+3x-4}{x^2-3x+2} = 1 + \frac{3x-4}{x^2-3x+2}.$$

Now, the denominator is $Q(x) = (x-1)(x-2)$, hence we set the partial fractions as

$$\frac{A}{x-1} + \frac{B}{x-2} = \frac{(A+B)x - (A+2B)}{x^2-3x+2}.$$

Hence, we need $A + B = 3$ and $A + 2B = 4$, which is given by $A = 1, B = 2$. Now we can integrate

$$\int \frac{x^2-2}{x^2-3x+2}dx = \int \left(1 + \frac{1}{x-1} + \frac{2}{x-2}\right)dx = x + \ln|x-1| + 2\ln|x-2| + C.$$

Practice Problems

Find the indefinite integral.

1. $\int \frac{1}{(x+3)(x-2)}dx$
2. $\int \frac{1}{x^2-4x+8}dx$
3. $\int \frac{1}{x^2+2x+10}dx$
4. $\int \frac{x-1}{x^2+7x+10}dx$
5. $\int \frac{1}{x^2-16}dx$
6. $\int \frac{1}{x^2+16}dx$

7. $\int \frac{1}{x^2-x-6} dx$

8. $\int \frac{x^2-4x-19}{x^2+5x+6} dx$

9. $\int \frac{x^3+2}{x^2+4} dx$

10. $\int \frac{x^2+9}{x^2-9} dx$

Solutions to Practice Problems

1. $\int \frac{1}{(x+3)(x-2)} dx$
   Partial fractions $\frac{1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2} = \frac{(A+B)x+3B-2A}{(x+3)(x-2)}$ give $A = -B = -1/5$, and hence we integrate as
$$\int \frac{1}{(x+3)(x-2)} dx = -\frac{1}{5}\left(\frac{1}{x+3} - \frac{1}{x-2}\right) = \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|$$

2. $\int \frac{1}{x^2-4x+8} dx$
   Complete the square: $x^2 - 4x + 8 = x^2 - 4x + 4 + 4 = (x-2)^2 + 4$, the integrate
$$\int \frac{dx}{x^2-4x+8} = \int \frac{dx}{(x-2)^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{x-2}{2}\right)^2+1} = \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\arctan\left(\frac{x-2}{2}\right)+C.$$

3. $\int \frac{1}{x^2+2x+10} dx$
   Complete the square: $x^2 + 2x + 10 = (x+1)^2 + 9$, then integrate
$$\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+9} = \frac{1}{9}\int \frac{dx}{\left(\frac{x+1}{3}\right)^2+1} = \frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C.$$

4. $\int \frac{x-1}{x^2+7x+10} dx$
   The denominator is $x^2 + 7x + 10 = (x+5)(x+2)$. Partial fractions $\frac{A}{x+5} + \frac{B}{x+2} = \frac{(A+B)x+2A+5B}{(x+5)(x+2)}$ give $A = 2, B = -1$ and then we integrate as
$$\int \frac{x-1}{x^2+7x+10} dx = \int \left(\frac{2}{x+5} - \frac{1}{x+2}\right) dx = 2\ln|x+5| - \ln|x+2| + C.$$

5. $\int \frac{1}{x^2-16} dx$
   The denominator is $x^2-16 = (x-4)(x+4)$. Partial fractions $\frac{A}{x-4} + \frac{B}{x+4} = \frac{(A+B)x+4(A-B)}{(x-4)(x+4)}$ give $A = -B = 1/8$ so that we can integrate as
$$\int \frac{1}{x^2-16} dx = \frac{1}{8}\int \left(\frac{1}{x-4} - \frac{1}{x+4}\right) dx = \frac{1}{8}\ln\left|\frac{x-4}{x+4}\right|.$$

6. $\int \frac{1}{x^2+16} dx$
   This can be integrated directly
$$\int \frac{1}{x^2+16} dx = \frac{1}{16}\int \frac{dx}{\left(\frac{x}{4}\right)^2+1} = \frac{1}{4}\arctan\left(\frac{x}{4}\right) + C.$$

7. $\int \frac{1}{x^2-x-6} dx$
   The denominator is $x^2-x-6 = (x-3)(x+2)$. Partial fractions $\frac{A}{x+2} + \frac{B}{x-3} = \frac{(A+B)x-3A+2B}{(x+2)(x-3)}$ give $A = -B = -1/5$ and then we integrate as
$$\int \frac{dx}{x^2-x-6} = -\frac{1}{5}\int \left(\frac{1}{x+2} - \frac{1}{x-3}\right) dx = \frac{1}{5}\ln\left|\frac{x-3}{x+2}\right| + C.$$

8. $\int \frac{x^2-4x-19}{x^2+5x+6}dx$
First of all long division, or write the numerator as $x^2+5x+6-9x-25$ to see that

$$\int \frac{x^2-4x-19}{x^2+5x+6}dx = \int \left(1 - \frac{9x+25}{x^2+5x+6}\right) dx.$$

Then do partial fractions with $x^2+5x+6 = (x+2)(x+3)$ and $\frac{A}{x+2}+\frac{B}{x+3} = \frac{(A+B)x+3A+2B}{(x+2)(x+3)}$ so that $A = 7$ and $B = 2$. The integrate as

$$\int \frac{x^2-4x-19}{x^2+5x+6}dx = \int \left(1 - \frac{7}{x+2} - \frac{2}{x+3}\right) dx = x - 7\ln|x+2| - 2\ln|x+3| + C.$$

9. $\int \frac{x^3+2}{x^2+4}dx$
Long division or rewrite the numerator as $x^3+2 = x(x^2+4)-4x+2$. Then (with the substitution $u = x^2+4$ in the second term)

$$\int \frac{x^3+2}{x^2+4}dx = \int \left(x - \frac{4x}{x^2+4} + \frac{2}{x^2+4}\right) dx = \frac{x^2}{2} - 2\int \frac{du}{u} + \frac{1}{2}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1} =$$

$$= \frac{x^2}{2} - 2\ln|x^2+4| + \arctan\left(\frac{x}{2}\right) + C.$$

10. $\int \frac{x^2+9}{x^2-9}dx$
First of all the numerator: $x^2+9 = x^2-9+18$ and the denominator $x^2-9 = (x-3)(x+3)$. This gives

$$\int \frac{x^2+9}{x^2-9}dx = \int \left(1 + \frac{18}{(x-3)(x+3)}\right) dx.$$

Then partial fractions $\frac{A}{x-3} + \frac{B}{x+3} = \frac{(A+B)x+3(A-B)}{(x-3)(x+3)}$. Hence $A = -B = 3$. Then we integrate as

$$\int \frac{x^2+9}{x^2-9}dx = \int \left(1 + \frac{3}{x-3} - \frac{3}{x+3}\right) dx = x + 3\ln\left|\frac{x-3}{x+3}\right| + C.$$